Spherical Trigonometry

The derivations behind Episode 2

In the film we used four relations without proving any of them. This is where they come from. Nothing below needs more than ordinary vector algebra and a little patience — every step is written out.

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1. Setting up

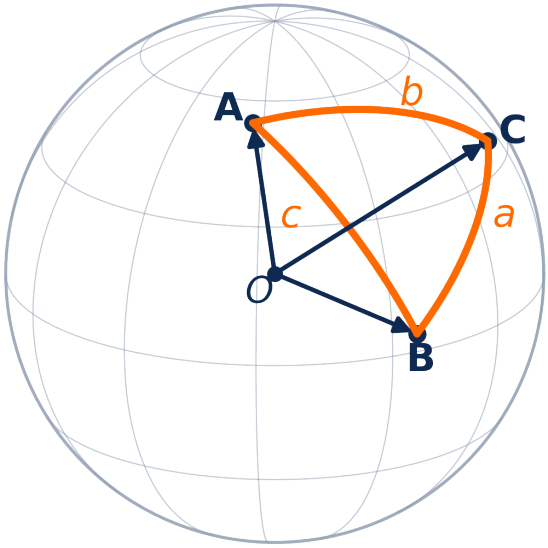
Put the centre of the sphere at the origin and let the radius be one. Each vertex of the triangle is then a unit vector:

$$\mathbf{A}, \qquad \mathbf{B}, \qquad \mathbf{C}, \qquad |\mathbf{A}| = |\mathbf{B}| = |\mathbf{C}| = 1.$$

A side of the triangle is the angle its two vertices subtend at the centre. Side a lies opposite vertex A, so it is the angle between $\mathbf{B}$ and $\mathbf{C}$:

$$\cos a = \mathbf{B}\cdot\mathbf{C}, \qquad \cos b = \mathbf{C}\cdot\mathbf{A}, \qquad \cos c = \mathbf{A}\cdot\mathbf{B}.$$

That single line is the bridge between geometry and algebra, and everything that follows rests on it.



The three vertices are unit vectors from the centre O. Each side of the triangle is the angle between two of them, so its cosine is simply their dot product.

The *angle* A is the spherical angle at vertex $\mathbf{A}$: the ordinary angle between the two tangents to the sides b and c where they leave $\mathbf{A}$. To write it down we need those tangents.

2. The tangent directions at a vertex

Step 1. Take the side c, which runs from $\mathbf{A}$ to $\mathbf{B}$. Its tangent at $\mathbf{A}$ points along $\mathbf{B}$ but must lie in the plane touching the sphere at $\mathbf{A}$ — that is, it must be perpendicular to $\mathbf{A}$. So we remove from $\mathbf{B}$ whatever part of it lies along $\mathbf{A}$:

$$\mathbf{u} = \mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{A}.$$

Step 2. How long is that vector? Since $\mathbf{A}\cdot\mathbf{B} = \cos c$,

$$|\mathbf{u}|^2 = \mathbf{B}\cdot\mathbf{B} - 2(\mathbf{A}\cdot\mathbf{B})^2 + (\mathbf{A}\cdot\mathbf{B})^2 = 1 - \cos^2 c = \sin^2 c,$$

so $|\mathbf{u}| = \sin c$, taking the positive root because every side of an Euler triangle lies strictly between 0° and 180°.

Step 3. The same for the side b, running from **A** to **C**:

$$\mathbf{v} = \mathbf{C} - (\mathbf{A}\cdot\mathbf{C})\,\mathbf{A}, \qquad |\mathbf{v}| = \sin b.$$

the tangent at **A**, in the plane of O, **A** and **B**

both tangents lie in the tangent plane at **A**



Left: removing from **B** the part that lies along **A** leaves **u**, perpendicular to **A** and of length $\sin c$. Right: **u** and **v** both lie in the plane touching the sphere at **A**, and the angle between them is the spherical angle A.

3. The law of cosines

Step 1. The angle A is the angle between **u** and **v**, so by the definition of the dot product

$$\cos A = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\sin b\,\sin c}.$$

Step 2. Expand the numerator, remembering $\mathbf{A}\cdot\mathbf{A} = 1$:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \big[\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{A}\big]\cdot\big[\mathbf{C} - (\mathbf{A}\cdot\mathbf{C})\mathbf{A}\big] \\ &= \mathbf{B}\cdot\mathbf{C} - (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{A}) - (\mathbf{A}\cdot\mathbf{B})(\mathbf{A}\cdot\mathbf{C}) + (\mathbf{A}\cdot\mathbf{B})(\mathbf{A}\cdot\mathbf{C}) \\ &= \mathbf{B}\cdot\mathbf{C} - (\mathbf{A}\cdot\mathbf{B})(\mathbf{A}\cdot\mathbf{C}) \;=\; \cos a - \cos c\,\cos b.\end{aligned}$$

Step 3. Put the two together:

$$\cos A = \frac{\cos a - \cos b\,\cos c}{\sin b\,\sin c},$$

and rearranging gives the relation as we used it in the film:

$$\boxed{\cos a = \cos b\,\cos c + \sin b\,\sin c\,\cos A}$$

Two sides and the angle between them on the right; the third side on the left. The same argument at the other two vertices gives the companion forms for $\cos b$ and $\cos c$.

4. The law of sines

Step 1. Start from the result just obtained and square it:

$$\sin^2 A = 1 - \cos^2 A = 1 - \frac{(\cos a - \cos b \cos c)^2}{\sin^2 b \, \sin^2 c}.$$

Step 2. Put everything over the common denominator and expand the numerator, writing $\sin^2 b = 1 - \cos^2 b$ and likewise for c:

$$\begin{aligned}\sin^2 b \, \sin^2 c - (\cos a - \cos b \cos c)^2 &= (1 - \cos^2 b)(1 - \cos^2 c) - \cos^2 a \\ &\quad + 2 \cos a \cos b \cos c - \cos^2 b \cos^2 c \\ &= 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c.\end{aligned}$$

Call that quantity K. Notice what has happened: every trace of which vertex we started from has vanished. K treats a, b and c alike.

Step 3. So

$$\frac{\sin^2 A}{\sin^2 a} = \frac{K}{\sin^2 a \, \sin^2 b \, \sin^2 c},$$

and the right-hand side is completely symmetric in the three sides. Had we begun at vertex B or C we would have reached the very same expression. Therefore

$$\frac{\sin^2 A}{\sin^2 a} = \frac{\sin^2 B}{\sin^2 b} = \frac{\sin^2 C}{\sin^2 c},$$

and taking positive roots — again safe, because no part of an Euler triangle reaches 180° — gives

$$\boxed{\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}}$$

The symmetry is the whole proof. Note also what the square root costs us: it fixes the magnitude of the angle but not its quadrant, which is exactly the warning given in the film.

5. The four-parts formula

We want a relation between four consecutive parts: the angle B, the side a, the angle C and the side b.

Step 1. Write the law of cosines twice, once for b and once for c:

$$\begin{aligned}\cos b &= \cos c \, \cos a + \sin c \, \sin a \, \cos B, \\ \cos c &= \cos a \, \cos b + \sin a \, \sin b \, \cos C.\end{aligned}$$

Step 2. Substitute the second line into the first, so that $\cos c$ disappears:

$$\cos b = \big(\cos a \cos b + \sin a \sin b \cos C\big) \cos a + \sin c \, \sin a \, \cos B.$$

Step 3. Collect the two terms in $\cos b$ on the left:

$$\cos b\,(1-\cos^2 a) = \sin a\,\sin b\,\cos a\,\cos C + \sin a\,\sin c\,\cos B,$$

and since $1-\cos^2 a = \sin^2 a$, divide throughout by $\sin a$:

$$\cos b\,\sin a = \sin b\,\cos a\,\cos C + \sin c\,\cos B.$$

Step 4. The term $\sin c$ still refers to a part we did not want. The law of sines removes it: from $\dfrac{\sin c}{\sin C} = \dfrac{\sin b}{\sin B}$ we get $\sin c = \dfrac{\sin b\,\sin C}{\sin B}$, and so

$$\sin c\,\cos B = \sin b\,\sin C\,\frac{\cos B}{\sin B} = \sin b\,\sin C\,\cot B.$$

Step 5. Substituting that back:

$$\cos b\,\sin a = \sin b\,\cos a\,\cos C + \sin b\,\sin C\,\cot B,$$

and dividing throughout by $\sin b$ leaves

$$\sin a\,\cot b = \cos a\,\cos C + \sin C\,\cot B,$$

which is the four-parts formula, usually written

$$\boxed{\cos a\,\cos C = \sin a\,\cot b - \sin C\,\cot B}$$

Unlike the sine rule, this one carries a cotangent, whose sign changes at 90°. That is precisely why it settles the quadrant, and why the azimuth was always worked from this relation rather than from the law of sines.

6. The haversine form

Step 1. Define the haversine and note the identity that makes it useful:

$$\operatorname{hav}\theta = \frac{1-\cos\theta}{2} = \sin^2\frac{\theta}{2}.$$

Step 2. Take the law of cosines and subtract both sides from one:

$$1-\cos a = 1-\cos b\,\cos c - \sin b\,\sin c\,\cos A.$$

Step 3. Here is the one trick in the whole derivation. Add and subtract $\sin b\sin c$ on the right:

$$1-\cos a = \big[\,1-(\cos b\cos c+\sin b\sin c)\,\big] + \sin b\,\sin c\,\big[\,1-\cos A\,\big].$$

Step 4. The bracket on the left is a cosine of a difference, $\cos(b-c) = \cos b\cos c+\sin b\sin c$, so

$$1-\cos a = \big[\,1-\cos(b-c)\,\big] + \sin b\,\sin c\,\big[\,1-\cos A\,\big].$$

Step 5. Divide by two, and every bracket becomes a haversine:

$$\boxed{\operatorname{hav} a = \operatorname{hav}(b-c) + \sin b \sin c \operatorname{hav} A}$$

No new mathematics was used — only the identity in Step 1 and a single add-and-subtract. The practical advantages of this arrangement are a story in themselves, and the films come back to them later.

7. Why one formula keeps the quadrant and the other does not

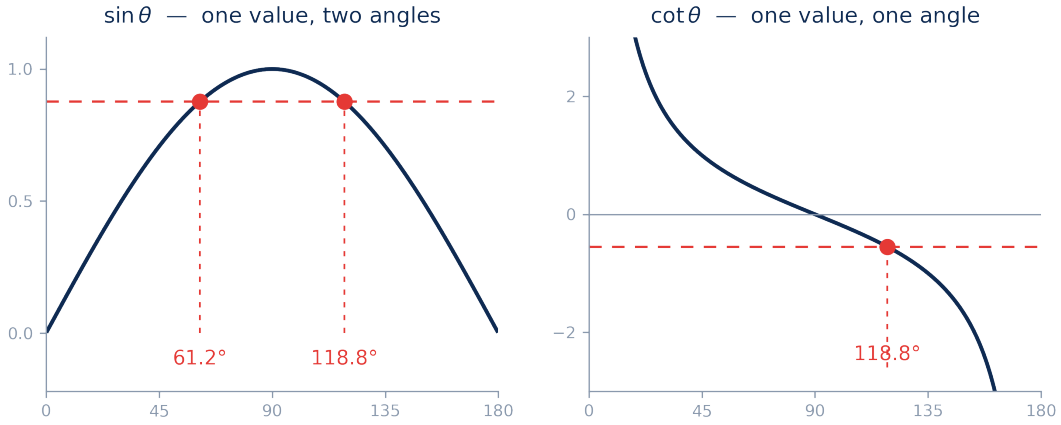
Solving a triangle means inverting a trigonometric function, and whether the answer is unique depends entirely on *which* function you end up inverting. Every part of an Euler triangle lies in $(0^\circ, 180^\circ)$, so that is the only interval we have to look at.

The sine. On $(0^\circ, 180^\circ)$ the sine is not one-to-one:

$$\sin(180^\circ - \theta) = \sin\theta.$$

Every value except 1 is taken twice, once on each side of 90°. So an equation of the form $\sin A = k$ has *two* solutions in range, and nothing in the equation itself says which one you want.

The cosine and the cotangent. Both are strictly decreasing across the whole interval — the cosine from $+1$ to -1, the cotangent from $+\infty$ to $-\infty$ — so each value is taken exactly once. An equation $\cos a = k$ or $\cot Z = k$ has a single solution, and its sign alone tells you which side of 90° you are on.



That is the whole difference. The law of cosines delivers a *cosine*, the four-parts formula delivers a *cotangent*; the law of sines delivers a sine, and a sine cannot distinguish θ from $180^\circ - \theta$.

A worked case where it matters

Take $\varphi = 50^\circ$ N, $\delta = 20^\circ$ N and LHA $= 40^\circ$. The altitude follows from the law of cosines:

$$\sin H_c = \sin 50^\circ \sin 20^\circ + \cos 50^\circ \cos 20^\circ \cos 40^\circ = 0.72471, \qquad H_c = 46.4^\circ.$$

Now the azimuth, first by the law of sines:

$$\sin Z = \frac{\cos\delta \sin \mathrm{LHA}}{\cos H_c} = \frac{0.93969 \times 0.64279}{0.68905} = 0.87660.$$

Taking the inverse sine on a calculator returns 61.2°. But 118.8° has exactly the same sine, and the equation cannot choose between them. Now the same azimuth from the four-parts formula, arranged for $\cot Z$:

$$\cot Z = \frac{\cos\varphi\tan\delta - \sin\varphi\cos\mathrm{LHA}}{\sin\mathrm{LHA}} = \frac{0.64279\times 0.36397 - 0.76604\times 0.76604}{0.64279} = -0.54897.$$

The value is negative, the cotangent is negative only beyond 90°, and the single solution is $Z = 118.8^\circ$. Checked directly against the geometry, that is the true azimuth — so the first answer the sine rule offered was not merely ambiguous, it was wrong.

That is why the azimuth was always worked from the four-parts formula, and why the sine rule, for all its elegance, is only ever used when something else has already settled the quadrant.

8. Back to the navigational triangle

The triangle PZX from Episode 1 is an ordinary spherical triangle, so the law of cosines applies to it directly. Substitute

$$a = 90^\circ - H_c, \qquad b = 90^\circ - \varphi, \qquad c = 90^\circ - \delta, \qquad A = \mathrm{LHA},$$

and use $\cos(90^\circ - x) = \sin x$ together with $\sin(90^\circ - x) = \cos x$:

$$\sin H_c = \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos\mathrm{LHA}.$$

That is the altitude formula at the heart of every sight reduction, and it is nothing more than the first relation in this document, wearing navigational clothes.

A note on signs: the substitution $b = 90^\circ - \varphi$ assumes latitude and declination of the same name. When they are contrary, the polar distance becomes $90^\circ + \delta$ and the sign of the second term reverses.

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